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Painlevé-Calogero Correspondence Revisited

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Abstract

We extend the work of Fuchs, Painlevé and Manin on a Calogero-like expression of the sixth Painlevé equation (the "Painlevé-Calogero correspondence") to the other five Painlevé equations. The Calogero side of the sixth Painlevé equation is known to be a non-autonomous version of the (rank one) elliptic model of Inozemtsev's extended Calogero systems. The fifth and fourth Painlevé equations correspond to the hyperbolic and rational models in Inozemtsev's classification. Those corresponding to the third, second and first are not included therein. We further extend the correspondence to the higher rank models, and obtain a "multi-component" version of the Painlevé equations.

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I Introduction

The so called Painlevé equations are the following six equations discovered by Painlevé [1] and Gambier [2]:

$$
\begin{aligned}
(\mathrm{P_{VI}})\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda-1}+\frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\
&+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\frac{\gamma(t-1)}{(\lambda-1)^2}+\frac{\delta t(t-1)}{(\lambda-t)^2}\right).\\
(\mathrm{P_{V}})\quad \frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda}+\frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}\\
&+\frac{\lambda(\lambda-1)^2}{t^2}\left(\alpha+\frac{\beta}{\lambda^2}+\frac{\gamma t}{(\lambda-1)^2}+\frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right).\\
(\mathrm{P_{IV}})\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2+\frac{3}{2}\lambda^3+4t\lambda^2+2(t^2-\alpha)\lambda+\frac{\beta}{\lambda}.\\
(\mathrm{P_{III}})\quad \frac{d^2\lambda}{dt^2} &= \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2-\frac{1}{t}\frac{d\lambda}{dt}+\frac{\lambda^2}{4t^2}\left(\alpha+\frac{\beta t}{\lambda^2}+\gamma\lambda+\frac{\delta t^2}{4\lambda^3}\right).\\
(\mathrm{P_{II}})\quad \frac{d^2\lambda}{dt^2} &= 2\lambda^3+t\lambda+\alpha.\\
(\mathrm{P_{I}})\quad \frac{d^2\lambda}{dt^2} &= 6\lambda^2+t.
\end{aligned}
$$

The third equation $\mathrm{P_{III}}$ is slightly modified; the original equation can be reproduced by the simple change of variables $(t,\lambda)\to(t^2,t\lambda)$. It is well known that these equations are characterized by the absence of "movable singularities" other than poles.

R. Fuchs [3] proposed two more approaches to the sixth equation $\mathrm{P_{VI}}$. One approach is the concept of isomonodromic deformations. In this approach, $\mathrm{P_{VI}}$ is interpreted as a differential equation describing isomonodromic deformations of a linear ordinary differential equation on the Riemann sphere. This is the origin of many subsequent researches. Another approach relates $\mathrm{P_{VI}}$ to an incomplete elliptic integral. Painlevé [4] took the second approach, and derived a new expression of $\mathrm{P_{VI}}$ in term of the Weierstrass $\wp$-function. This work of Painlevé is briefly reviewed in Okamoto's work on affine Weyl group symmetries of $\mathrm{P_{VI}}$ [5].

Manin [6] revived the almost forgotten work of Fuchs and Painlevé after nearly ninety years. Manin's remarkable idea is to use the elliptic modulus τ, rather than t, as an independent variable. The outcome is a Hamiltonian system with a Hamiltonian of the normal form $\mathcal{H}=p^2/2+V(q)$, where the potential is a linear combination of the Weier-

strass $\wp$-function and its shift by three three half periods. This is a non-autonomous system, because the Hamiltonian depends on the "time" τ through the τ-dependence of the $\wp$-function.

Levin and Olshanetsky [7] pointed out that Manin's equation resembles the so called Calogero-Moser systems, i.e., the various extensions [8] of the integrable many-body systems first discovered by Calogero [9]. More precisely, the Hamiltonian $\mathcal{H}$ is identical to a special case (the rank-one elliptic model) of Inozemtsev's extensions [10, 11] of the Calogero-Moser systems. Levin and Olshanetsky called this relation the "Painlevé-Calogero correspondence".

One will naturally ask if this correspondence can be extended to the other Painlevé equations. Manin himself raised this problem in his paper. Olshanetsky [12] conjectured that a degenerate version of Inozemtsev's elliptic model will emerge therein.

This paper aims to answer this question affirmatively. A guiding principle is the degeneration relation of the six Painlevé equations [13]. This relation can be schematically expressed as follows:

$$\begin{array}{ccccccc} \mathrm{P_{VI}} & \longrightarrow & \mathrm{P_{V}} & \longrightarrow & \mathrm{P_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_{I}} \end{array}$$

This diagram means, for instance, that $\mathrm{P_V}$ can be derived from $\mathrm{P_{VI}}$ by a degeneration process, which amounts to confluence of singular points of the aforementioned linear ordinary differential equation in the isomonodromic approach. We shall trace this process carefully on the "Calogero side", and find a $\mathrm{P_V}$-version of Manin's equation. In principle, one can thus find an analogue of Manin's equation for all the six Painlevé equations (though, actually, one can resort to a more direct approach that bypasses the complicated degeneration process).

Remarkably (or rather naturally?), all the six equations on the Calogero side turn out to become a (non-autonomous) Hamiltonian system with a Hamiltonian of the normal form $\mathcal{H} = p^2/2 + V(q)$. Furthermore, the Hamiltonians on the Calogero side of $\mathrm{P_V}$ and $\mathrm{P_{IV}}$ coincide with the Hamiltonians of the (rank one) hyperbolic and rational models in Inozemtsev's classification [10] (which were discovered by Levi and Wojciechowski [14] before Inozemtsev's work). Those corresponding to the other three Painlevé equations are

not included therein, but may be thought of as a further degeneration of the hyperbolic and rational models.

One can further proceed to the higher rank models, and ask if there is still a Painlevé-Calogero correspondence. We shall show that this is also the case. The Painlevé side of the correspondence is a kind of multi-dimensional extensions of the Painlevé equations. They are obviously different from another multi-dimensional extension called the "Garnier systems" [13]. For this reason, we call our multi-dimensional extension a *multi-component* version of the Painlevé equations.

This paper is organized as follows. Section 2 is a brief review of the work of Fuchs, Painlevé and Manin. Section 3 deals with $\mathrm{P_V}$, $\mathrm{P_{IV}}$ and $\mathrm{P_{III}}$. The degeneration process is discussed in detail for the case of $\mathrm{P_V}$. The direct approach is illustrated for the case of $\mathrm{P_{IV}}$ and $\mathrm{P_{III}}$. Section 4 shows a reformulation of the foregoing calculations in the Hamiltonian formalism. The status of $\mathrm{P_{II}}$ and $\mathrm{P_I}$ is also clarified therein. Section 5 is devoted to the higher rank Inozemtsev Hamiltonians and the multi-component Painlevé equations. Section 6 is for concluding remarks. Part of technical details are gathered in Appendices.

II Painlevé-Calogero Correspondence for $\mathrm{P_{VI}}$

We here briefly review the work of Fuchs, Painlevé and Manin.

Fuchs rewrites $\mathrm{P_{VI}}$ into the following form:

$$\begin{aligned} & t(1-t)\mathcal{L}_t \int_\infty^\lambda \frac{dz}{\sqrt{z(z-1)(z-t)}} \\ = \; & \sqrt{\lambda(\lambda-1)(\lambda-t)} \left[\alpha + \frac{\beta t}{\lambda^2} + \frac{\gamma(t-1)}{(\lambda-1)^2} + \left(\delta - \frac{1}{2}\right) \frac{t(t-1)}{(\lambda-t)^2} \right]. \end{aligned} \tag{1}$$

Here $\mathcal{L}_t$ is the linear differential operator (Picard-Fuchs operator)

$$\mathcal{L}_t = t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}, \tag{2}$$

which also appears in the Picard-Fuchs equation of complete elliptic integrals. In this respect, $\mathrm{P_{VI}}$ may be thought of as an inhomogeneous (and nonlinear) analogue of the Picard-Fuchs equation.

Painlevé and Manin make use of a parametrization of the elliptic curve

$$y^2 = z(z-1)(z-t) \tag{3}$$

by the Weierstrass $\wp$-function. Let $\wp(u)$ be the $\wp$-function with primitive periods 1 and τ:

$$\wp(u) = \wp(u \mid 1, \tau) = \frac{1}{u^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(u+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right), \tag{4}$$

The parametrization is now given by

$$z = \frac{\wp(u) - e_1}{e_2 - e_1}, \quad y = \frac{\wp'(u)}{2(e_2 - e_1)^{3/2}}, \tag{5}$$

where $e_n = \wp(\omega_n)$, $n = 1, 2, 3$ are the values of $\wp(u)$ at the three half period points $\omega_1 = 1/2$, $\omega_2 = -(1+\tau)/2$, $\omega_3 = \tau/2$.

Manin's excellent idea is to do a simultaneous change of the dependent variable $\lambda \to q$ by

$$\lambda = \frac{\wp(q) - e_1}{e_2 - e_1}, \tag{6}$$

and the independent vrariable $t \to \tau$ by

$$t = \frac{e_3 - e_1}{e_2 - e_1}. \tag{7}$$

Manin presents the beautiful formula

$$\frac{d\tau}{dt} = \frac{\pi i}{t(t-1)(e_2 - e_1)}, \tag{8}$$

for the Jacobian of the latter, which plays a key role in his calculations. $\mathrm{P_{VI}}$ is thereby mapped to the equation

$$(2\pi i)^2 \frac{d^2 q}{d\tau^2} = \sum_{n=0}^{3} \alpha_n \wp'(q + \omega_n), \tag{9}$$

where the parameters on the right hand side are connected with the parameters of $\mathrm{P_{VI}}$ as $\alpha_0 = \alpha$, $\alpha_1 = -\beta$, $\alpha_2 = \gamma$, $\alpha_3 = -\delta + 1/2$. This equation is equivalent to the Hamiltonian system

$$2\pi i \frac{dq}{d\tau} = \frac{\partial \mathcal{H}}{\partial p}, \quad 2\pi i \frac{dp}{d\tau} = -\frac{\partial \mathcal{H}}{\partial q} \tag{10}$$

with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \sum_{n=0}^{3} \alpha_n \wp(q + \omega_n). \tag{11}$$

III Correspondence for $\mathrm{P_V}$, $\mathrm{P_{IV}}$ and $\mathrm{P_{III}}$

III.1 Degeneration of $\mathrm{P_{VI}}$ to $\mathrm{P_V}$

The degeneration of $\mathrm{P_{VI}}$ to $\mathrm{P_V}$ is achieved by rescaling the time variable and the parameters as

$$t = 1 + \epsilon \tilde{t}, \quad \alpha = \tilde{\alpha}, \quad \beta = \tilde{\beta}, \quad \gamma = \frac{\tilde{\gamma}}{\epsilon} - \frac{\tilde{\delta}}{\epsilon^2}, \quad \delta = \frac{\tilde{\delta}}{\epsilon^2} \tag{12}$$

and letting $\epsilon \to 0$ while leaving $\tilde{\alpha}, \ldots, \tilde{\gamma}$ and $\tilde{t}$ finite [13].

The building blocks of Fuchs' equation (1) turn out to survive this scaling limit as follows:

1. The Picard-Fuchs operator:
$$t(1-t)\mathcal{L}_t \longrightarrow \tilde{t}^2 \frac{d^2}{d\tilde{t}^2} + \tilde{t}\frac{d}{d\tilde{t}} = \left(\tilde{t}\frac{d}{d\tilde{t}}\right)^2.$$
2. The sum $\alpha + \cdots$ of four terms on the right hand side:
$$\alpha + \frac{\beta t}{\lambda^2} + \frac{\gamma(t-1)}{(\lambda-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{t(t-1)}{(\lambda-t)^2} \longrightarrow \tilde{\alpha} + \frac{\tilde{\beta}}{\lambda^2} + \frac{\tilde{\gamma}\tilde{t}}{(\lambda-1)^2} + \frac{\tilde{\delta}\tilde{t}^2(\lambda+1)}{(\lambda-1)^3}.$$
3. The quare root on the right hand side:
$$\sqrt{\lambda(\lambda-1)(\lambda-t)} \longrightarrow \sqrt{\lambda}(\lambda-1).$$
4. The incomplete elliptic integral:
$$\int_\infty^\lambda \frac{dz}{\sqrt{z(z-1)(z-t)}} \longrightarrow \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)}.$$

In particular, the degeneration of $\mathrm{P_{VI}}$ to $\mathrm{P_V}$ is associated with the degeneration of the elliptic curve to a rational curve,

$$y^2 = z(z-1)(z-t) \quad \longrightarrow \quad y^2 = z(z-1)^2, \tag{13}$$

or, equivalently, the degeneration of the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ to the cylinder $\mathbb{C}/\mathbb{Z}$.

Thus, rewriting $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ and $\tilde{t}$ to $\alpha, \beta, \gamma, \delta$ and t, we obtain the following equation as a $\mathrm{P_V}$-version of Fuchs' equation:

$$\left(t\frac{d}{dt}\right)^2 \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)} = \sqrt{\lambda}(\lambda-1)\left(\alpha + \frac{\beta}{\lambda^2} + \frac{\gamma t}{(\lambda-1)^2} + \frac{\delta t^2(\lambda+1)}{(\lambda-1)^3}\right). \tag{14}$$

III.2 Analogue of Manin's equation for $\mathrm{P_V}$

As an counterpart of the q-variable for $\mathrm{P_{VI}}$, we now consider

$$q = \int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)}. \tag{15}$$

If one prefers to being more faithful to Manin's parametrization, one should rather define q as

$$q = \frac{1}{2\pi i}\int_\infty^\lambda \frac{dz}{\sqrt{z}(z-1)},$$

because $2(e_2 - e_1)^{1/2} \to 2\pi i$ as $\mathrm{Im}\,\tau \to +\infty$ (see Appendix B). Since there is no substantial difference, let us take the first definition that is slightly simpler for calculations.

Let us rewrite (14) in terms of q. The integral can be readily calculated as

$$q = \log\left(\frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}\right), \tag{16}$$

so that the inverse relation can be written

$$\sqrt{\lambda} = -\coth(q/2). \tag{17}$$

Terms on the right hand side of (14) can be calculated as follows:

$$\begin{aligned}
\sqrt{\lambda}(\lambda-1) &= -\frac{\cosh(q/2)}{\sinh^3(q/2)}, \\
\sqrt{\lambda}(\lambda-1)\frac{1}{\lambda^2} &= -\frac{\sinh(q/2)}{\cosh^3(q/2)}, \\
\sqrt{\lambda}(\lambda-1)\frac{1}{(\lambda-1)^2} &= -\frac{1}{2}\sinh(q), \\
\sqrt{\lambda}(\lambda-1)\frac{(\lambda+1)}{(\lambda-1)^3} &= -\frac{\lambda^{3/2}+\lambda^{1/2}}{(\lambda-1)^2} = -\frac{1}{4}\sinh(2q).
\end{aligned}$$

The differential equation for q eventually takes the form

$$\left(t\frac{d}{dt}\right)^2 q = -\frac{\partial V(q)}{\partial q}, \tag{18}$$

where

$$V(q) = -\frac{\alpha}{\sinh^2(q/2)} - \frac{\beta}{\cosh^2(q/2)} + \frac{\gamma t}{2}\cosh(q) + \frac{\delta t^2}{8}\cosh(2q). \tag{19}$$

This gives a $\mathrm{P_V}$-version of Manin's equation. Note that this equation can be readily converted to a Hamiltonian system with the Hamiltonian $\mathcal{H} = p^2/2 + V(q)$.

Remark.

A very similar change of dependent variable for $\mathrm{P_V}$ is discussed in the book of Iwasaki et al. [15].

III.3 Idea of direct approach

Although the degeneration process can be continued to the other Painlevé equations, we now present a more direct approach. Note that the integrand is connected with the coefficient of $(d\lambda/dt)^2$ in the original Painlevé equation by the following very simple relation:

$$\begin{aligned}
\frac{1}{\sqrt{z(z-1)(z-t)}} &= \exp\left[-\int \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-t}\right) dz\right], \\
\frac{1}{\sqrt{z}(z-1)} &= \exp\left[-\int \left(\frac{1}{2z} + \frac{1}{z-1}\right) dz\right].
\end{aligned}$$

If this is a correct prescription, one will be able to define the q-variable for $\mathrm{P_{III}}$ and $\mathrm{P_{II}}$ directly without the cumbersome degeneration process. This is indeed the case, as we shall show below.

III.4 q-variable for $\mathrm{P_{IV}}$

Since the expected integrand is given by

$$\exp\left(-\int \frac{dz}{2z}\right) = \frac{1}{\sqrt{z}}, \tag{20}$$

we define

$$q = \int^{\lambda} \frac{dz}{\sqrt{z}} = 2\sqrt{\lambda}. \tag{21}$$

This can be solved for λ as

$$\lambda = \left(\frac{q}{2}\right)^2. \tag{22}$$

Honest calculations show that all derivative terms of P_{IV} can be absorbed by the second derivative of q:

$$\begin{aligned} \frac{d^2q}{dt^2} &= \frac{1}{\sqrt{\lambda}}\frac{d^2\lambda}{dt^2} - \frac{1}{2\lambda\sqrt{\lambda}}\left(\frac{d\lambda}{dt}\right)^2 \\ &= \frac{1}{\sqrt{\lambda}}\left(\frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2-\alpha)\lambda + \frac{\beta}{\lambda}\right). \end{aligned} \tag{23}$$

Substituting $\lambda = (q/2)^2$ gives the second order differential equation

$$\frac{d^2q}{dt^2} = -\frac{\partial V(q)}{\partial q} \tag{24}$$

with the potential

$$V(q) = -\frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2-\alpha)\left(\frac{q}{2}\right)^2 + \beta\left(\frac{q}{2}\right)^{-2}. \tag{25}$$

III.5 q-variable for $\mathrm{P}_{\mathrm{III}}$

The integrand is expected to be given by

$$\exp\left(-\int\frac{dz}{z}\right) = \frac{1}{z}. \tag{26}$$

We consider

$$q = \int^{\lambda}\frac{dz}{z} = \log\lambda \tag{27}$$

and its inversion

$$\lambda = e^q. \tag{28}$$

All derivatives terms of $\mathrm{P}_{\mathrm{III}}$ are now absorbed by the second derivative of q with respect to $\log t$:

$$\begin{aligned} \left(t\frac{d}{dt}\right)^2 q &= \frac{t^2}{\lambda}\frac{d^2\lambda}{dt^2} + \frac{t}{\lambda}\frac{d\lambda}{dt} - \frac{t^2}{\lambda^2}\left(\frac{d\lambda}{dt}\right)^2 \\ &= \frac{\alpha\lambda}{4} + \frac{\beta t}{4\lambda} + \frac{\gamma\lambda^2}{4} + \frac{\delta t^2}{4\lambda^2}. \end{aligned} \tag{29}$$

Substituting $\lambda = e^q$ gives the second order equation

$$\left(t\frac{d}{dt}\right)^2 q = -\frac{\partial V(q)}{\partial q} \tag{30}$$

with the potential

$$V(q) = -\frac{\alpha}{4}e^q + \frac{\beta t}{4}e^{-q} - \frac{\gamma}{8}e^{2q} + \frac{\delta t^2}{8}e^{-2q}. \tag{31}$$

III.6 Summary

Let us summarize the results of this section.

Theorem 1 *The foregoing change of variable $\lambda \to q$ maps $\mathrm{P_V}$, $\mathrm{P_{IV}}$ and $\mathrm{P_{III}}$ to a second order differential equation for the new dependent variable q. These equations are equivalent to a non-autonomous Hamiltonian system with a Hamiltonian of the normal form $\mathcal{H} = p^2/2 + V(q)$:*

($\mathrm{P_V}$) *The Hamiltonian system takes the form*

$$t\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad t\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{32}$$

with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{\alpha}{\sinh^2(q/2)} - \frac{\beta}{\cosh^2(q/2)} + \frac{\gamma t}{2}\cosh(q) + \frac{\delta t^2}{8}\cosh(2q). \tag{33}$$

($\mathrm{P_{IV}}$) *The Hamiltonian system takes the form*

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{34}$$

with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2 - \alpha)\left(\frac{q}{2}\right)^2 + \beta\left(\frac{q}{2}\right)^{-2}. \tag{35}$$

($\mathrm{P_{III}}$) *The Hamiltonian system takes the form*

$$t\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad t\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{36}$$

with the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} - \frac{\alpha}{4}e^{q} + \frac{\beta t}{4}e^{-q} - \frac{\gamma}{8}e^{2q} + \frac{\delta t^2}{8}e^{-2q}. \tag{37}$$

Remark.

1. The Hamiltonians for $\mathrm{P_V}$ and $\mathrm{P_{IV}}$ coincide with those of the hyperbolic and rational models of Inozemtsev [10], Levi and Wojciechowski [14]. The Hamiltonian for $\mathrm{P_{III}}$ has no counterpart in their work, but nowadays can be found in the literature [16].

2. The foregoing construction of the q-variable does not literally work for $\mathrm{P_{II}}$ and $\mathrm{P_I}$, because there is no $(d\lambda/dt)^2$ term. The status of these equations will be clarified in the next section from a different point of view.

IV Hamiltonian formalism of correspondence

IV.1 Hamiltonians of Painlevé equations

All the six Painlevé equations are known to be expressed in the Hamiltonian form

$$\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\lambda}{dt} = -\frac{\partial H}{\partial \lambda}$$

with a suitable choice of the canonical conjugate variable μ and the Hamiltonian H [18]. This expression is by no means unique; we here consider the following Hamiltonians [13]. These Hamiltonians are referred to as the "polynomial Hamiltonians" because they are polynomials in λ and μ:

$$(\mathrm{P_{VI}}) \quad H = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right)\mu + \frac{\kappa}{\lambda(\lambda-1)}\right].$$

$$(\mathrm{P_{V}}) \quad H = \frac{\lambda(\lambda-1)^2}{t}\left[\mu^2 - \left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\eta_1 t}{(\lambda-1)^2}\right)\mu + \frac{\kappa}{\lambda(\lambda-1)}\right].$$

$$(\mathrm{P_{IV}}) \quad H = 2\lambda\left[\mu^2 - \left(\frac{\lambda}{2} + t + \frac{\kappa_0}{\lambda}\right)\mu + \frac{\theta_\infty}{2}\right].$$

$$(\mathrm{P_{III}}) \quad H = \frac{\lambda^2}{t}\left[\mu^2 - \left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)\mu + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2\lambda}\right].$$

$$(\mathrm{P_{II}}) \quad H = \frac{\mu^2}{2} - \left(\lambda^2 + \frac{t}{2}\right)\mu - \left(\alpha + \frac{1}{2}\right)\lambda.$$

$$(\mathrm{P_{I}}) \quad H = \frac{\mu^2}{2} - 2\lambda^3 - t\lambda.$$

Here $\kappa_0, \kappa_1, \theta$, etc. are constants that are connected with the parameters $\alpha, \beta, \gamma, \delta$ of the Painlevé equations by simple algebraic relations:

$$(\mathrm{P_{VI}}) \quad \alpha = \frac{(\kappa_0+\kappa_1+\theta-1)^2}{2} - 2\kappa,\ \beta = -\frac{\kappa_0^2}{2},\ \gamma = \frac{\kappa_1^2}{2},\ \delta = \frac{1-\theta^2}{2},$$

$$(\mathrm{P_{V}}) \quad \alpha = \frac{(\kappa_0+\theta_1)^2}{2} - 2\kappa,\ \beta = -\frac{\kappa_0^2}{2},\ \gamma = \eta_1(\theta_1+1),\ \delta = -\frac{\eta_1^2}{2}.$$

$$(\mathrm{P_{IV}}) \quad \alpha = 2\theta_\infty - \kappa_0 + 1,\ \beta = -2\kappa_0^2.$$

$$(\mathrm{P_{III}}) \quad \alpha = -4\eta_\infty\theta_\infty,\ \beta = 4\eta_0(\theta_0+1),\ \gamma = 4\eta_\infty^2,\ \delta = -4\eta_0^2.$$

IV.2 How to find canonical transformations

The goal of this section is to show that the Painlevé-Calogero correspondence is, in fact, a (time-dependent) canonical transformation of two Hamiltonian systems. By this, we

mean that the functional relation between λ and q can be extended to (λ, μ) and (q, p) so as to satisfy the equation

$$\mu d\lambda - Hdt = \text{constant} \cdot (pdq - \mathcal{H}dT) + \text{exact form.} \tag{38}$$

with a suitably redefined time variable T (such as the logarithmic time $\log t$ in $\mathrm{P_V}$ and $\mathrm{P_{III}}$). The constant factor on the right hand side is inserted simply for convenience; if necessary, one can normalize the constant to 1 by suitably rescaling $p, q, \mathcal{H}$ and T. For this reason, wel call this type of coordinate transformation a "canonical" transformation even if the constant factor is not equal to 1.

Let us illustrate, in the case of $\mathrm{P_{VI}}$, how to find such a canonical transformation. Suppose that λ and μ be a solution of $\mathrm{P_{VI}}$ in the aforementioned Hamiltonian formalism, and that q be a corresponding solution of Manin's equation. The canonical equation for λ takes the form

$$\frac{d\lambda}{dt} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \left(2\mu - \frac{\kappa_0}{\lambda} - \frac{\kappa_1}{\lambda-1} - \frac{\theta-1}{\lambda-t}\right).$$

This equations can be solve for μ:

$$\mu = \frac{t(t-1)}{2\lambda(\lambda-1)(\lambda-t)} \frac{d\lambda}{dt} + \frac{1}{2}\left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right).$$

Our task is to rewrite the right hand side in terms of p and q. We first consider $d\lambda/dt$. Differentiating (6) against t gives

$$\frac{d\lambda}{dt} = \left(\frac{\wp'(q)}{e_2-e_1}\frac{dq}{d\tau} + f_\tau(q)\right)\frac{d\tau}{dt},$$

where we have introduced the functions

$$f(u) = \frac{\wp(u)-e_1}{e_2-e_1}, \quad f_\tau(u) = \frac{\partial f(u)}{\partial \tau}. \tag{39}$$

The derivative $dq/d\tau$ can be read off from the canonical equation for q:

$$\frac{dq}{d\tau} = \frac{1}{2\pi i}\frac{\partial \mathcal{H}}{\partial p} = \frac{p}{2\pi i}.$$

As for the Jacobian $d\tau/dt$, Manin's formula (8) is available. One can thus express $d\lambda/dt$ as a function of p, q and τ. The other part of the foregoing expression of μ contains λ

only, which can be readily converted to a function of q and τ by (6). We thus obtain the following expression of μ:

$$
\begin{aligned}
\mu &= \frac{e_2 - e_1}{\wp'(q)} p + \frac{2\pi i (e_2 - e_1)^2}{\wp'(q)^2} f_\tau(q) \\
&\quad + \frac{e_2 - e_1}{2} \left(\frac{\kappa_0}{\wp(q) - e_1} + \frac{\kappa_1}{\wp(q) - e_2} + \frac{\theta - 1}{\wp(q) - e_3} \right).
\end{aligned} \tag{40}
$$

We now move the point of view, and think of (6) and (40) as defining a coordinate transformation $(\lambda, \mu) \to (q, p)$. This gives a canonical transformation that we have sought for:

Theorem 2 *(6) and (40) define a canonical transformation that connects the Hamiltonian form of* $\mathrm{P_{VI}}$ *and Manin's Hamiltonian system. The canonical coordinates and the Hamiltonians of the two systems obey the equation*

$$
\mu d\lambda - H dt = p dq - \mathcal{H} \frac{d\tau}{2\pi i} + \text{exact form}. \tag{41}
$$

IV.3 Proof of Theorem 2

Total differential of (6) gives

$$
d\lambda = \frac{\wp'(q)}{e_2 - e_1} dq + f_\tau(q) d\tau,
$$

so that $\mu d\lambda$ can be expressed as

$$
\begin{aligned}
\mu d\lambda &= \left(\frac{e_2 - e_1}{\wp'(q)} p + \frac{2\pi i (e_2 - e_1)^2}{\wp'(q)^2} f_\tau(q) \right) \left(\frac{\wp'(q)}{e_2 - e_1} dq + f_\tau(q) d\tau \right) \\
&\quad + \frac{1}{2} \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda - 1} + \frac{\theta - 1}{\lambda - t} \right) d\lambda \\
&= p dq + \text{(A)} + \text{(B)} + \text{(C)},
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(A)} &= \frac{2\pi i (e_2 - e_1)}{\wp'(q)} f_\tau(q) dq, \\
\text{(B)} &= \left(\frac{e_2 - e_1}{\wp'(q)} p + \frac{2\pi i (e_2 - e_1)^2}{\wp'(q)^2} f_\tau(q) \right) f_\tau(q) d\tau, \\
\text{(C)} &= \frac{1}{2} \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda - 1} + \frac{\theta - 1}{\lambda - t} \right) d\lambda.
\end{aligned}
$$

As we shall prove in Appendix A, (A) can be further rewritten

$$(\mathrm{A}) = \left[\frac{\wp(q+\omega_3)}{4\pi i} - \pi\left(\frac{f_\tau(q)}{f'(q)}\right)^2\right] d\tau + \text{exact form}, \tag{42}$$

where $f'(u)$ denotes the u-derivative of $f(u)$:

$$f'(u) = \frac{\partial f(u)}{\partial u} = \frac{\wp'(u)}{e_2 - e_1}. \tag{43}$$

For (B) and (C), we have

$$\begin{aligned}
(\mathrm{B}) &= \left[\frac{f_\tau(q)}{f'(q)}p + 2\pi i\left(\frac{f_\tau(q)}{f'(q)}\right)^2\right] d\tau,\\
(\mathrm{C}) &= \frac{\theta-1}{2(\lambda-t)}dt + \frac{1}{2}\left(\kappa_0\log\lambda + \kappa_1\log(\lambda-1) + (\theta-1)\log(\lambda-t)\right)\\
&= \frac{\theta-1}{2(\lambda-t)}dt + \text{exact form.}
\end{aligned}$$

Thus we find that

$$\mu d\lambda - H dt = p dq - \tilde{\mathcal{H}}\frac{d\tau}{2\pi i} + \text{exact form}, \tag{44}$$

where

$$\tilde{\mathcal{H}} = 2\pi i\frac{dt}{d\tau}\left(H - \frac{\theta-1}{2(\lambda-t)}\right) - 2\pi i\left[\frac{\wp(q+\omega_3)}{4\pi i} + \frac{f_\tau(q)}{f'(q)}p + \pi i\left(\frac{f_\tau(q)}{f'(q)}\right)^2\right]. \tag{45}$$

Our task is to prove that the transformed Hamiltonian $\tilde{\mathcal{H}}$ coincides, modulo irrelevant terms, with the Hamiltonian of Manin's equation. Here "irrelevant" means that the term is a function of t only. Such a "non-dynamical" term can be absorbed by the "exact form" part of the foregoing relation of 1-forms, thereby being negligible.

Let us evaluate the contribution of $2\pi i(dt/d\tau)H$. By Manin's formula (8) of $d\tau/dt$, and also by the identity

$$\lambda(\lambda-1)(\lambda-t) = \frac{\wp'(q)^2}{4(e_2-e_1)^3},$$

we can rewrite $2\pi i(dt/d\tau)H$ as follows:

$$\begin{aligned}
2\pi i\frac{dt}{d\tau}H &= \frac{\wp'(q)^2}{2(e_2-e_1)^2}\left[\mu^2 - \left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right)\mu + \frac{\kappa}{\lambda(\lambda-1)}\right]\\
&= \frac{\wp'(q)^2}{2(e_2-e_1)^2}\left[\mu - \frac{1}{2}\left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right)\right]^2\\
&\quad + \frac{\wp'(q)^2}{2(e_2-e_1)^2}\left[-\frac{1}{4}\left(\frac{\kappa_0}{\lambda} + \frac{\kappa_1}{\lambda-1} + \frac{\theta-1}{\lambda-t}\right)^2 + \frac{\kappa}{\lambda(\lambda-1)}\right].
\end{aligned}$$

The first term on the right hand side is equal to

$$\frac{1}{2}\left(p+2\pi i\frac{f_\tau(q)}{f'(q)}\right)^2=\frac{p^2}{2}+2\pi i\frac{f_\tau(q)}{f'(q)}p+\left(2\pi i\frac{f_\tau(q)}{f'(q)}\right)^2,$$

by which the terms proportional to $f_\tau(q)/f'(q)$ and its square in the definition of $\tilde{\mathcal{H}}$ are cancelled out. The transformed Hamiltonian $\tilde{\mathcal{H}}$ can now be expressed as

$$\begin{aligned}\tilde{\mathcal{H}} &= \frac{p^2}{2}-\frac{\wp'(q)^2}{2(e_2-e_1)^2}-\frac{(\theta-1)t(t-1)(e_2-e_1)}{\lambda-t}\\ &+\frac{\wp'(q)}{2(e_2-e_1)^2}\left[-\frac{1}{4}\left(\frac{\kappa_0}{\lambda}+\frac{\kappa_1}{\lambda-1}+\frac{\theta-1}{\lambda-t}\right)^2+\frac{\kappa}{\lambda(\lambda-1)}\right].\end{aligned}\tag{46}$$

Note that this is already of the normal form $p^2/2+\tilde{V}(q)$ with the potential

$$\begin{aligned}\tilde{V}(q) &= -\frac{\wp'(q)^2}{2(e_2-e_1)^2}-\frac{(\theta-1)t(t-1)(e_2-e_1)}{\lambda-t}\\ &+\frac{\wp'(q)}{2(e_2-e_1)^2}\left[-\frac{1}{4}\left(\frac{\kappa_0}{\lambda}+\frac{\kappa_1}{\lambda-1}+\frac{\theta-1}{\lambda-t}\right)^2+\frac{\kappa}{\lambda(\lambda-1)}\right].\end{aligned}\tag{47}$$

What remains is to express $\tilde{V}(q)$ as an explicit function of q. To this end, we substitute the factor $\wp'(q)^2/2(e_2-e_1)^2$ by $2(e_2-e_1)\lambda(\lambda-1)(\lambda-t)$, and rewrite the main part of $\tilde{V}(q)$ as a linear combination of λ, $1/\lambda$, $1/(\lambda-1)$ and $1/(\lambda-t)$. This leads to the following expression of $\tilde{V}(q)$:

$$\begin{aligned}\tilde{V}(q) &= -\frac{(\kappa_0+\kappa_1+\theta-1)^2-4\kappa}{2}(e_2-e_1)\lambda\\ &-\frac{\kappa_0^2}{2}\cdot\frac{(e_2-e_1)t}{\lambda}-\frac{\kappa_1^2}{2}\cdot\frac{(e_2-e_1)(1-t)}{\lambda-1}-\frac{(\theta-1)^2+1}{2}\cdot\frac{(e_2-e_1)t(t-1)}{\lambda-t}\\ &-\frac{1}{2}\wp(q+\omega_3)+\text{function of } t \text{ only.}\end{aligned}$$

The final piece of the ring is the general formula

$$\wp(u+\omega_j)=e_j+\frac{(e_j-e_k)(e_j-e_\ell)}{\wp(u)-e_j}\tag{48}$$

where (j,k,l) is a cyclic permutation of $(1,2,3)$. This implies that

$$\begin{aligned}\frac{(e_2-e_1)t}{\lambda} &= \wp(q+\omega_1)-e_1,\\ \frac{(e_2-e_1)(1-t)}{\lambda-1} &= \wp(q+\omega_2)-e_2,\\ \frac{(e_2-e_1)t(t-1)}{\lambda-t} &= \wp(q+\omega_3)-e_3,\end{aligned}$$

so that

$$\tilde{V}(q) = -\frac{(\kappa_0+\kappa_1+\theta-1)^2-4\kappa}{2}\wp(q)-\frac{\kappa_0^2}{2}\wp(q+\omega_1)$$
$$-\frac{\kappa_1^2}{2}\wp(q+\omega_2)-\frac{\theta^2}{2}\wp(q+\omega_3)+\text{function of } \tau \text{ only.} \tag{49}$$

Apart from the last term which is negligible, this potential is indeed the same as Manin's potential $V(q)$ (recall the algebraic relations connecting the constants κ_0, etc. and the parameters of $\mathrm{P_{VI}}$). This completes the proof of the theorem. *Q.E.D.*

IV.4 Canonical transformation for $\mathrm{P_V}$

This heuristic method for constructing a canonical transformation can be applied to the other Painlevé equations. We here consider the case of $\mathrm{P_V}$.

Let λ be a solution of $\mathrm{P_V}$, μ the canonical conjugate variable, and q the corresponding solution of (18). The canonical equation for λ can be written

$$\frac{d\lambda}{dt} = \frac{\lambda(\lambda-1)^2}{t}\left(2\mu-\frac{\kappa_0}{\lambda}-\frac{\theta_1}{\lambda-1}+\frac{\eta_1 t}{(\lambda-1)^2}\right).$$

This equation can be solved for μ as

$$\mu = \frac{1}{2\lambda(\lambda-1)^2}t\frac{d\lambda}{dt}+\frac{1}{2}\left(\frac{\kappa_0}{\lambda}+\frac{\theta_1}{\lambda-1}-\frac{\eta_1 t}{(\lambda-1)^2}\right).$$

By differentiating (17) against t and using the canonical equation $tdq/dt = \partial\mathcal{H}/\partial p = p$, we obtain the identity

$$t\frac{d\lambda}{dt} = \sqrt{\lambda}(\lambda-1)p,$$

which can be used to rewrite the expression of μ as

$$\mu = \frac{p}{2\sqrt{\lambda}(\lambda-1)}+\frac{1}{2}\left(\frac{\kappa_0}{\lambda}+\frac{\theta_1}{\lambda-1}-\frac{\eta_1 t}{(\lambda-1)^2}\right). \tag{50}$$

We now reinterpret (17) and (50) as defining a coordinate transformation $(\lambda,\mu)\to(q,p)$. This indeed turns out to give a canonical transformation that we have sought for:

Theorem 3 *(17) and (50) define a canonical transformation that connects* $\mathrm{P_V}$ *and the* $\mathrm{P_V}$*-version of Manin's Hamiltonian system. The canonical coordinates and the Hamiltonians of the two systems obey the equation*

$$\mu d\lambda - H dt = \frac{1}{2}\left(pdq-\mathcal{H}\frac{dt}{t}\right)+\text{exact form.} \tag{51}$$

Proof. Since $d\lambda$ and dq are connected by the relation

$$d\lambda = \sqrt{\lambda}(\lambda-1)dq,$$

$\mu d\lambda$ can be expressed as

$$\begin{aligned}\mu d\lambda &= \frac{1}{2}pdq + \frac{1}{2}\left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\eta_1 t}{(\lambda-1)^2}\right)d\lambda \\ &= \frac{1}{2}pdq - \frac{\eta_1}{2(\lambda-1)}dt + \frac{1}{2}d\left(\kappa_0 \log\lambda + \theta_1\log(\lambda-1) + \frac{\eta_1 t}{\lambda-1}\right),\end{aligned}$$

so that

$$\mu d\lambda - Hdt = \frac{1}{2}\left(pdq - \tilde{\mathcal{H}}\frac{dt}{t}\right) + \text{exact form}, \tag{52}$$

where

$$\tilde{\mathcal{H}} = 2Ht + \frac{\eta_1 t}{\lambda-1}. \tag{53}$$

We can rewrite $\tilde{\mathcal{H}}$ to a normal form as

$$\begin{aligned}\tilde{\mathcal{H}} &= 2\lambda(\lambda-1)^2\left[\mu - \frac{1}{2}\left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\theta_1 t}{(\lambda-1)^2}\right)\right]^2 \\ &\quad + 2\lambda(\lambda-1)^2\left[-\frac{1}{4}\left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\eta_1 t}{(\lambda-1)^2}\right)^2 + \frac{\kappa}{\lambda(\lambda-1)}\right] + \frac{\eta_1 t}{\lambda-1} \\ &= \frac{p^2}{2} + \tilde{V}(q),\end{aligned} \tag{54}$$

where

$$\begin{aligned}\tilde{V}(q) &= -\frac{\lambda(\lambda-1)^2}{2}\left(\frac{\kappa_0}{\lambda} + \frac{\theta_1}{\lambda-1} - \frac{\eta_1 t}{(\lambda-1)^2}\right)^2 + 2\kappa(\lambda-1) + \frac{\eta_1 t}{\lambda-1}. \\ &= -\left(\frac{\kappa_0}{2} + \frac{\theta_1^2}{2} + \kappa_1\theta_1 - 2\kappa\right)\frac{1}{\sinh^2(q/2)} + \frac{\kappa_0^2}{2}\frac{1}{\cosh^2(q/2)} \\ &\quad + \frac{\eta_1(\theta_1+1)t}{2}\cosh(q) - \frac{\eta_1^2 t^2}{2}\cosh(2q) + \text{function of } t \text{ only}.\end{aligned} \tag{55}$$

Apart from the last negligible term, this coincides with the potential $V(q)$ in the statement of the theorem. *Q.E.D.*

IV.5 Canonical transformation for $\mathrm{P_{IV}}$

We now consider the case of $\mathrm{P_{IV}}$.

Let λ be a solution of $\mathrm{P_{IV}}$, μ the canonical conjugate variable, and q the corresponding solution of (24). The canonical equation for λ can be written

$$\frac{d\lambda}{dt} = 4\lambda\mu - (\lambda^2 + 2t\lambda + 2\kappa_0),$$

which can be solved for μ as

$$\mu = \frac{1}{4\lambda}\frac{d\lambda}{dt} + \frac{1}{4}\left(\lambda + 2t + \frac{2\kappa_0}{\lambda}\right).$$

By (22) and the canonical equation $dq/dt = \partial\mathcal{H}/\partial p = p$, we have the identity

$$\frac{d\lambda}{dt} = \sqrt{\lambda}\frac{dq}{dt} = \sqrt{\lambda}p,$$

so that

$$\mu = \frac{p}{4\sqrt{\lambda}} + \frac{1}{4}\left(\lambda + 2t + \frac{2\kappa_0}{\lambda}\right). \tag{56}$$

Theorem 4 *(22) and (56) define a canonical transformation that connects* $\mathrm{P_{IV}}$ *and the* $\mathrm{P_{IV}}$*-version of Manin's Hamiltonian system. The canonical coordinates and Hamiltonians of the two systems obey the equation*

$$\mu d\lambda - H dt = \frac{1}{4}(pdq - \mathcal{H}dt) + \text{exact form}. \tag{57}$$

Proof. Since $d\lambda$ and dq are connected by the relation

$$d\lambda = \sqrt{\lambda}dq,$$

$\mu d\lambda$ can be expressed as

$$\begin{aligned}\mu d\lambda &= \frac{1}{4}pdq + \frac{1}{4}\left(\lambda + 2t + \frac{2\kappa_0}{\lambda}\right)d\lambda \\ &= \frac{1}{4}pdq - \frac{1}{2}\lambda dt + \frac{1}{4}d\left(\frac{\lambda^2}{2} + 2t\lambda + 2\kappa_0\log\lambda\right),\end{aligned}$$

so that

$$\mu d\lambda - H dt = \frac{1}{4}(pdq - \tilde{\mathcal{H}}dt) + \text{exact form}, \tag{58}$$

where

$$\tilde{\mathcal{H}} = 4H + 2\lambda. \tag{59}$$

We can rewrite the transformed Hamiltonian $\tilde{\mathcal{H}}$ to a normal form as

$$\begin{aligned}
\tilde{\mathcal{H}} &= 8\lambda\left[\mu - \frac{1}{2}\left(\frac{\lambda}{2} + t + \frac{\kappa_0}{\lambda}\right)\right]^2 + 8\lambda\left[-\frac{1}{4}\left(\frac{\lambda}{2} + t + \frac{\kappa_0}{\lambda}\right)^2 + \frac{\theta_\infty}{2}\right] + 2\lambda \\
&= \frac{p^2}{2} + \tilde{V}(q),
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
\tilde{V}(q) &= -2\lambda\left(\frac{\lambda}{2} + t + \frac{\kappa_0}{\lambda}\right)^2 + 4\theta_\infty\lambda + 2\lambda \\
&= -\frac{1}{2}\lambda^3 - 2t\lambda^2 - 2(t^2 + \kappa_0 - 2\theta_\infty - 1)\lambda - 2\kappa_0^2\lambda^{-1} \\
&\quad + \text{function of } t \text{ only.}
\end{aligned} \tag{61}$$

Substituting $\lambda = (q/2)^2$ gives the potential $V(q)$ modulo an irrelevant term. *Q.E.D.*

IV.6 Canonical transformations for $\mathrm{P_{III}}$

The situation of $\mathrm{P_{III}}$ is somewhat similar to $\mathrm{P_V}$.

Let λ, again, be a solution of $\mathrm{P_{III}}$, λ the canonical conjuage variable, and q be the corresponding solution of (30). The canonical equation for λ takes the form

$$\frac{d\lambda}{dt} = \frac{\lambda^2}{t}\left(2\mu - \eta_\infty - \frac{\theta_0}{\lambda} + \frac{\eta_0 t}{\lambda^2}\right),$$

which can be solved for μ as

$$\mu = \frac{t}{2\lambda^2}\frac{d\lambda}{dt} + \frac{1}{2}\left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right).$$

By differentiating (28) and using the canonical equation $tdq/dt = \partial\mathcal{H}/\partial p = p$, the t-derivative of λ can be written

$$t\frac{d\lambda}{dt} = \lambda p,$$

so that we obtain

$$\mu = \frac{p}{2\lambda} + \frac{1}{2}\left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right). \tag{62}$$

This relation, again, can be used to define a canonical transformation:

Theorem 5 *(28) and (62) define a canonical transformation that connects* $\mathrm{P_{III}}$ *and the* $\mathrm{P_{III}}$*-version of Manin's Hamiltonian system. The canonical coordinates and the Hamiltonians of the two systems obey the equation*

$$\mu d\lambda - Hdt = \frac{1}{2}\left(pdq - \mathcal{H}\frac{dt}{t}\right) + \text{exact form}. \tag{63}$$

Proof. Since $d\lambda$ and dq are connected by the relation

$$d\lambda = \lambda dq,$$

$\mu d\lambda$ can be written

$$\begin{aligned} \mu d\lambda &= \frac{1}{2}pdq + \frac{1}{2}\left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)d\lambda \\ &= \frac{1}{2}pdq - \frac{\eta_0}{2\lambda}dt + \frac{1}{2}d\left(\eta_\infty\lambda + \theta_0\log\lambda + \frac{\eta_0 t}{\lambda}\right), \end{aligned}$$

so that

$$\mu d\lambda - Hdt = \frac{1}{2}\left(pdq - \tilde{\mathcal{H}}\frac{dt}{t}\right) + \text{exact form}, \tag{64}$$

where

$$\tilde{\mathcal{H}} = 2Ht + \frac{\eta_0 t}{\lambda}. \tag{65}$$

We can convert the transformed Hamiltonian $\tilde{\mathcal{H}}$ to a normal form as

$$\begin{aligned} \tilde{\mathcal{H}} &= 2\lambda^2\left[\mu - \frac{1}{2}\left(\eta_\infty + \frac{\eta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)\right]^2 \\ &\quad + 2\lambda^2\left[-\frac{1}{2}\left(\eta_\infty + \frac{\eta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)^2 + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2\lambda}\right] + \frac{\eta_0 t}{\lambda} \\ &= \frac{p^2}{2} + \tilde{V}(q), \end{aligned} \tag{66}$$

where

$$\begin{aligned} \tilde{V}(q) &= -\frac{\lambda^2}{2}\left(\eta_\infty + \frac{\theta_0}{\lambda} - \frac{\eta_0 t}{\lambda^2}\right)^2 + \eta_\infty(\theta_0+\theta_\infty)\lambda + \frac{\eta_0 t}{\lambda} \\ &= \eta_\infty\theta_\infty e^q + \eta_0(\theta_0+1)te^{-q} - \frac{\eta_\infty^2}{2}e^{2q} - \frac{\eta_0^2 t^2}{2}e^{-2q} \\ &\quad + \text{function of } t \text{ only}. \end{aligned} \tag{67}$$

Thus, apart from the last irrelevant term, $\tilde{V}(q)$ coincides with the potential $V(q)$ in the statement of the theorem. *Q.E.D.*

IV.7 Status of $\mathrm{P_{II}}$ and $\mathrm{P_I}$

Let us turn to $\mathrm{P_{II}}$ and $\mathrm{P_I}$. The Hamiltonian of $\mathrm{P_I}$ is already of the normal form $\mathcal{H} = \frac{p^2}{2} + V(q)$ with $\lambda = q$, $\mu = p$ and $H = \mathcal{H}$. Although this is not the case for $\mathrm{P_{II}}$, one can directly find a canonical transformation that converts the Hamiltonian H to a normal form:

Theorem 6 *A $\mathrm{P_{II}}$-version of Manin's Hamiltonian system is defined by the Hamiltonian*

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \alpha q. \tag{68}$$

This system is connected with $\mathrm{P_{II}}$ by the canonical transformation

$$\lambda = q, \quad \mu = p + \lambda^2 + \frac{t}{2}. \tag{69}$$

The canonical coordinates and the Hamiltonians of the two systems obey the equation

$$\mu d\lambda - H dt = p dq - \mathcal{H} dt + \text{exact form}. \tag{70}$$

Proof. The foregoing relation between (λ, μ) and (q, p) implies that

$$\mu d\lambda = p dq + \left(\lambda^2 + \frac{t}{2}\right) d\lambda = p dq - \frac{\lambda}{2} dt + d\left(\frac{\lambda^3}{3} + \frac{t\lambda}{2}\right),$$

so that

$$\mu d\lambda - H dt = p dq - \tilde{\mathcal{H}} dt + \text{exact form}, \tag{71}$$

where

$$\begin{aligned} \tilde{\mathcal{H}} &= H + \frac{\lambda}{2} \\ &= \frac{1}{2}\left[\mu - \left(\lambda^2 + \frac{t}{2}\right)\right]^2 - \frac{1}{2}\left(\lambda^2 + \frac{t}{2}\right)^2 - \left(\alpha + \frac{1}{2}\right)\lambda + \frac{\lambda}{2} \\ &= \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \alpha q. \end{aligned} \tag{72}$$

This is nothing but the Hamiltonian in the statement of the theorem. *Q.E.D.*

V Multi-component Painlevé equations

V.1 Inozemtsev Hamiltonians of higher rank

The rank ℓ version of Inozemtsev's Hamiltonians have ℓ coordinates $q_1, \ldots, q_\ell$ and canonical conjugate momenta $p_1, \ldots, p_\ell$. The Hamiltonians of the elliptic, hyperbolic and rational models take the following form [10, 11, 14]:

- Elliptic model:

$$\mathcal{H} = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + \sum_{n=0}^{3} g_n^2 \wp(q_j + \omega_n) \right) + g_4^2 \sum_{j \neq k} \left(\wp(q_j - q_k) + \wp(q_j + q_k) \right).$$

- Hyperbolic model:

$$\begin{aligned} \mathcal{H} &= \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + \frac{g_0^2}{\sinh^2(q_j/2)} + \frac{g_1^2}{\cosh^2(q_j/2)} + g_2^2 \cosh(q_j) + g_3^2 \cosh(2q_j) \right) \\ &\quad + g_4^2 \sum_{j \neq k} \left(\frac{1}{\sinh^2((q_j - q_k)/2)} + \frac{1}{\sinh^2((q_j + q_k)/2)} \right). \end{aligned}$$

- Rational model:

$$\mathcal{H} = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + g_0^2 q_j^6 + g_1^2 q_j^4 + g_2^2 q_j^2 + g_3^2 q_j^{-2} \right) + g_4^2 \sum_{j \neq k} \left(\frac{1}{(q_j - q_k)^2} + \frac{1}{(q_j + q_k)^2} \right).$$

Here g_0, g_1, g_2, g_3 and g_4 are coupling constants. The Painlevé-Calogero correspondence for $\mathrm{P_{III}}$, $\mathrm{P_{II}}$ and $\mathrm{P_I}$ suggests the existence of further degeneration of these models.

The goal of this section is to extend the the Painlevé-Calogero correspondence to these higher rank models. Since a complete exposition will become inevitably lengthy, we shall illustrate the elliptic and hyperbolic models in detail, leaving the other cases rather sketchy. The strategy is as follows: The point of departure is the Hamiltonian of Inozemtsev's rank ℓ elliptic model. This gives rise to a rank ℓ version of Manin's equation. Starting with this non-autonomous Hamiltonian system, we seek for an analogue of the degeneration process for the Painlevé equations. We can thus obtain six types of non-autonomous Hamiltonian systems. At each stage of the degeneration process, we confirm that the non-autonomous Hamiltonian system on the Calogero side can be mapped, by a canonical transformation, to a multi-component analogue of the Painlevé equation of the corresponding type.

V.2 Elliptic model and multi-component $\mathrm{P_{VI}}$

We now consider the non-autonomous Hamiltonian system

$$2\pi i\frac{dq_j}{d\tau} = \frac{\partial\mathcal{H}}{\partial p_j}, \quad 2\pi i\frac{dp_j}{d\tau} = -\frac{\partial\mathcal{H}}{\partial q_j} \tag{73}$$

defined by the Hamiltonian of Inozemtsev's elliptic model. This is a rank ℓ version of Manin's equation. This non-autonomous system is known to describe a family of isomonodromic deformations on the torus [17].

An honest generalization of the canonical transformation for the case of $\ell = 1$ leads to a multi-component version of $\mathrm{P_{VI}}$ as follows:

Theorem 7 *The time-dependent canonical transformation defined by*

$$\begin{aligned}
\lambda_j &= \frac{\wp(q_j) - e_1}{e_2 - e_1}, \\
\mu_j &= \frac{e_2 - e_1}{\wp'(q)}p_j + \frac{2\pi i(e_2 - e_1)^2}{\wp'(q_j)^2}f_\tau(q_j) \\
&\quad + \frac{e_2 - e_1}{2}\left(\frac{\kappa_0}{\wp(q_j) - e_1} + \frac{\kappa_1}{\wp(q_j) - e_2} + \frac{\theta - 1}{\wp(q_j) - e_3}\right),
\end{aligned} \tag{74}$$

and

$$t = \frac{e_3 - e_1}{e_2 - e_1}. \tag{75}$$

maps (73) to the Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \tag{76}$$

with the Hamiltonian

$$\begin{aligned}
H &= \sum_{j=1}^{\ell} \frac{\lambda_j(\lambda_j - 1)(\lambda_j - t)}{t(t-1)}\left[\mu_j^2 - \left(\frac{\kappa_0}{\lambda_j} + \frac{\kappa_1}{\lambda_j - 1} + \frac{\theta - 1}{\lambda_j - t}\right)\mu_j + \frac{\kappa}{\lambda_j(\lambda_j - 1)}\right] \\
&\quad + \frac{g_4^2}{2t(t-1)}\sum_{j\neq k}\left[\frac{\lambda_j(\lambda_j - 1)(\lambda_j - t) + \lambda_k(\lambda_k - 1)(\lambda_k - t)}{8(\lambda_j - \lambda_k)^2} - 2(\lambda_j + \lambda_k)\right].
\end{aligned} \tag{77}$$

Proof. The method of proof for the case of $\ell = 1$ can be applied to the present case as well, yielding the equality

$$\sum_{j=1}^{\ell} p_j dq_j - \mathcal{H}\frac{d\tau}{2\pi i} = \sum_{j=1}^{\ell}\mu_j d\lambda_j - \tilde{H}dt + \text{exact form}, \tag{78}$$

where

$$\begin{aligned}\tilde{H} &= \sum_{j=1}^{\ell} \frac{\lambda_j(\lambda_j-1)(\lambda_j-t)}{t(t-1)} \left[\mu_j^2 - \left(\frac{\kappa_0}{\lambda_j} + \frac{\kappa_1}{\lambda_j - 1} + \frac{\theta-1}{\lambda_j - t}\right)\mu_j + \frac{\kappa}{\lambda_j(\lambda_j-1)}\right] \\ &\quad + \frac{g_4^2}{2t(t-1)(e_2-e_1)} \sum_{j\neq k} \Big(\wp(q_j-q_k) + \wp(q_j+q_k)\Big). \end{aligned} \tag{79}$$

What remains is to express the "two-body potential" part in terms of λ_j. To this end, let us recall the addition formula

$$\wp(u-v) + \wp(u+v) = -2\wp(u) - 2\wp(v) + \frac{\wp'(u)^2 + \wp'(v)^2}{2(\wp(u)-\wp(v))^2} \tag{80}$$

of the $\wp$-function. Applying it to the case where $(u,v) = (\lambda_j, \lambda_k)$, and substituting

$$\begin{aligned}\wp(q_j) &= e_1 + (e_2-e_1)\lambda_j, \\ \wp(q_k) &= e_1 + (e_2-e_1)\lambda_k, \\ \wp'(q_j)^2 &= \frac{(e_2-e_1)^3}{4}\lambda_j(\lambda_j-1)(\lambda_j-t), \\ \wp'(q_k)^2 &= \frac{(e_2-e_1)^3}{4}\lambda_k(\lambda_k-1)(\lambda_k-t),\end{aligned}$$

we can rewrite the two-body potential terms as

$$\begin{aligned}\wp(q_j-q_k) + \wp(q_j+q_k) &= -2(e_1+(e_2-e_1)\lambda_j) - 2(e_1+(e_2-e_1)\lambda_k) \\ &\quad + \frac{(e_2-e_1)^3}{8} \cdot \frac{\lambda_j(\lambda_j-1)(\lambda_j-t) + \lambda_k(\lambda_k-1)(\lambda_k-t)}{(e_1+(e_2-e_1)\lambda_j - e_1 - (e_2-e_1)\lambda_k)^2} \\ &= -4e_1 - 2(e_2-e_1)(\lambda_j+\lambda_k) \\ &\quad + \frac{e_2-e_1}{8} \cdot \frac{\lambda_j(\lambda_j-1)(\lambda_j-t) + \lambda_k(\lambda_k-1)(\lambda_k-t)}{(\lambda_j-\lambda_k)^2}. \end{aligned} \tag{81}$$

The first term $-4e_1$ is non-dynamical, thereby negligible (i.e., can be absorbed by the "exact form" part). Removing these terms from $\tilde{H}$, we obtain the Hamiltonian H. *Q.E.D.*

V.3 Degeneration of elliptic model to hyperbolic model

The degeneration of the elliptic model is achieved by letting $\mathrm{Im}\,\tau \to +\infty$. Like the degeneration process from $\mathrm{P_{VI}}$ to $\mathrm{P_V}$, this is a kind of scaling limit, namely, the coupling constants g_n and the elliptic modulus τ have to be suitably rescaled. To this end, we have

to understand the asymptotic behavior of the constants e_1, e_2, e_3 and the $\wp$-function in the limit as $\operatorname{Im}\tau \to +\infty$. All necessary data are collected in Appendix B. For instance, the asymptotic expression of e_1, e_2 and e_3 imply that

$$t = 1 + \frac{e_3 - e_2}{e_2 - e_1} = 1 + 16\pi^2 e^{\pi i\tau} + O(e^{2\pi i\tau}). \tag{82}$$

This is indeed consistent with the scaling rule $t = 1 + \epsilon\tilde{t}$ in the degeneration process of $\mathrm{P_{VI}}$ to $\mathrm{P_V}$.

Having these data, we now rescale the coupling constants and the elliptic modulus as

$$g_0^2 = \tilde{g}_0^2, \quad g_1^2 = \tilde{g}_1^2, \quad g_2^2 = \frac{\tilde{g}_2^2}{\epsilon} + \frac{\tilde{g}_3^2}{\epsilon^2}, \quad g_3^3 = \frac{\tilde{g}_3^2}{\epsilon^2}, \quad g_4^2 = \tilde{g}_4^2 \tag{83}$$

and

$$16 e^{\pi i\tau} = \epsilon\tilde{t}, \tag{84}$$

and consider the limit as $\epsilon \to 0$ while leaving $\tilde{g}_n$ and $\tilde{t}$ finite. Note that letting $\epsilon \to 0$ amounts to letting $\operatorname{Im}\tau \to +\infty$.

The asymptotic expression of $\wp(u)$ and $\wp(u+\omega_n)$ in Appendix B show that the potential $V(q)$ of the elliptic model behaves as

$$\begin{aligned} V(q) \;\; = \;\; & \sum_{j=1}^{\ell}\left(\frac{\tilde{g}_0^2\pi^2}{\sin^2(\pi q_j)} + \frac{\tilde{g}_1^2\pi^2}{\cos^2(\pi q_j)} + \frac{\tilde{g}_2^2\pi^2\tilde{t}}{2}\cos(2\pi q_j) - \frac{\tilde{g}_3^2\pi^2\tilde{t}^2}{8}\cos(4\pi q_j)\right) \\ & + \tilde{g}_4^2 \sum_{j\neq k}\left(\frac{1}{\sin^2(\pi(q_j - q_k))} + \frac{1}{\sin^2(\pi(q_j + q_k))}\right) \\ & + \text{function of } \epsilon \text{ and } \tilde{t} \text{ only} + O(\epsilon). \end{aligned}$$

Thus, removing negligible terms, we obtain the following Hamiltonian in the limit:

$$\begin{aligned} \tilde{\mathcal{H}} \;\; = \;\; & \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2} + \frac{\tilde{g}_0^2\pi^2}{\sin^2(\pi q_j)} + \frac{\tilde{g}_1^2\pi^2}{\cos^2(\pi q_j)} + \frac{\tilde{g}_2^2\pi^2\tilde{t}}{2}\cos(2\pi q_j) - \frac{\tilde{g}_3^2\pi^2\tilde{t}^2}{8}\cos(4\pi q_j)\right) \\ & + \tilde{g}_4^2 \sum_{j\neq k}\left(\frac{1}{\sin^2(\pi(q_j - q_k))} + \frac{1}{\sin^2(\pi(q_j + q_k))}\right). \end{aligned} \tag{85}$$

The asymptotic expression of t determines the equation of motion in the limit. In fact, since

$$\frac{d\tau}{dt} = \frac{\pi}{t(t-1)(e_2 - e_1)} = \frac{\pi i}{(1+\epsilon\tilde{t})(-\epsilon\tilde{t})(-\pi^2 + O(\epsilon))}$$

and

$$2\pi i \frac{d}{d\tau} = 2\pi i \frac{dt}{d\tau}\frac{d\tilde{t}}{dt}\frac{d}{dt} = \left(2\pi^2 \tilde{t} + O(\epsilon^2)\right)\frac{d}{dt},$$

we find that the equations of motion take the following form:

$$2\pi^2 \tilde{t}\frac{dq_j}{d\tilde{t}} = \frac{\partial \tilde{\mathcal{H}}}{\partial p_j}, \quad 2\pi^2 \tilde{t}\frac{dp_j}{d\tilde{t}} = -\frac{\partial \tilde{\mathcal{H}}}{\partial q_j}. \tag{86}$$

The final step is to rescale the variables and the Hamiltonian as

$$q_j \to \frac{q_j}{2\pi i}, \quad p_j \to \pi i q_j, \quad \tilde{\mathcal{H}} \to -\pi^2 \tilde{\mathcal{H}}, \tag{87}$$

and to rename $\tilde{t}$ and $\tilde{\mathcal{H}}$ to t and $\mathcal{H}$. Let us also define the new constants

$$\alpha = -\frac{\tilde{g}_0^2}{2}, \quad \beta = \frac{\tilde{g}_1^2}{2}, \quad \gamma = -\frac{\tilde{g}_2^2}{2}, \quad \delta = \frac{\tilde{g}_3^2}{2}, \tag{88}$$

which are to be identified with the four parameters of $\mathrm{P_V}$. The outcome is the non-autonomous Hamiltonian system

$$t\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad t\frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{89}$$

with the Hamiltonian

$$\begin{aligned} \mathcal{H} \ = \ & \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2} - \frac{\alpha}{\sinh^2(q_j/2)} - \frac{\beta}{\cosh^2(q_j/2)} + \frac{\gamma t}{2}\cosh(q_j) + \frac{\delta t^2}{8}\cosh(2q_j)\right) \\ & + g_4^2 \sum_{j\neq k}\left(\frac{1}{\sinh^2((q_j-q_k)/2)} + \frac{1}{\sinh^2((q_j+q_k)/2)}\right). \end{aligned} \tag{90}$$

This gives a rank ℓ version of the non-autonomous Hamiltonian system on the Calogero side of $\mathrm{P_V}$. Note that the Hamiltoian is essentially the same as the Hamiltonian of Inozemtsev's hyperbolic model, except that the effective coupling constants are now time-dependent.

Remark. The foregoing prescription of scaling limit of the coupling constants and the elliptic modulus is reminiscent of "renormalization" in quantum field theories. In this analogy, one can interpret the equations of motion of the Hamiltonian system as "renormalization group equations", in which $\tilde{t}$ plays the role of a "mass scale" parameter.

V.4 Canonical transformation to multi-component $\mathrm{P_V}$

Again, an honest generalization of the canonical transformation for the case of $\ell = 1$ leads to a multi-component version of $\mathrm{P_V}$:

Theorem 8 *The time-dependent canonical transformation defined by*

$$\begin{aligned} \sqrt{\lambda_j} &= -\coth(q_j/2), \\ \mu_j &= \frac{p_j}{2\sqrt{\lambda_j}(\lambda_j - 1)} + \frac{1}{2}\left(\frac{\kappa_0}{\lambda_j} + \frac{\theta_1}{\lambda_j - 1} - \frac{\eta_1 t}{(\lambda_j - 1)^2}\right) \end{aligned} \tag{91}$$

maps (89) to the Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \tag{92}$$

with the Hamiltonian

$$\begin{aligned} H &= \sum_{j=1}^{\ell} \frac{\lambda_j(\lambda_j - 1)^2}{t}\left[\mu_j^2 - \left(\frac{\kappa_0}{\lambda_j} + \frac{\theta_1}{\lambda_j - 1} - \frac{\eta_1 t}{(\lambda_j - 1)^2}\right)\mu_j + \frac{\kappa}{\lambda_j(\lambda_j - 1)}\right] \\ &\quad + \frac{g_4^2}{2t}\sum_{j \neq k} \frac{2(\lambda_j - 1)(\lambda_k - 1)(\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k)^2}. \end{aligned} \tag{93}$$

Proof. The method of proof for the case of $\ell = 1$ can be used as it is. The outcome is the equality

$$\sum_{j=1}^{\ell} p_j dq_j - \mathcal{H}\frac{dt}{t} = 2\left(\sum_{j=1}^{\ell} \mu_j d\lambda_j - H dt\right) + \text{exact form}, \tag{94}$$

where

$$\begin{aligned} H &= \sum_{j=1}^{\ell} \frac{\lambda_j(\lambda_j - 1)^2}{t}\left[\mu_j^2 - \left(\frac{\kappa_0}{\lambda_j} + \frac{\theta_1}{\lambda_j - 1} - \frac{\eta_1 t}{(\lambda_j - 1)^2}\right)\mu_j + \frac{\kappa}{\lambda_j(\lambda_j - 1)}\right] \\ &\quad + \frac{g_4^2}{2t}\sum_{j \neq k}\left(\frac{1}{\sinh^2((q_j - q_k)/2)} + \frac{1}{\sinh^2((q_j + q_k)/2)}\right). \end{aligned} \tag{95}$$

The two-body potential part can be rewritten by use of the identity

$$\frac{1}{\sinh^2(u - v)} + \frac{1}{\sinh^2(u + v)} = 4\frac{\cosh(2u)\cosh(2v) - 1}{(\cosh(2u) - \cosh(2v))^2}. \tag{96}$$

Substituting $u = q_j/2$, $v = q_k/2$, and also using the equality $\cosh(q_j) = (\lambda_j + 1)/(\lambda_j - 1)$, we find that

$$\frac{1}{\sinh^2((q_j - q_k)/2)} + \frac{1}{\sinh^2((q_j + q_k)/2)} = \frac{2(\lambda_j - 1)(\lambda_k - 1)(\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k)^2}, \tag{97}$$

which gives the two-body potential term in H. *Q.E.D.*

V.5 Other models

The degeneration process can be further continued, and leads to four more models that correspond to a multi-component version of P_{IV}, P_{III}, P_{II} and P_I. Since the details of derivation are more or less parallel, we show the final results only. The Hamiltonian of each model, like those in the foregoing cases, becomes a sum of ℓ copies of the one-component Hamiltonian and Calogero-like two-body potential terms.

V.5.1 Rational model and multi-component P_{IV}

This model can be derived from the hyperbolic model by degeneration. The degeneration process consists of putting the variables and the parameters as

$$t = 1 + 2\epsilon\tilde{t}, \quad q_j = \pi i + \epsilon^{1/2}\tilde{q}_j, \quad p_j = \frac{\tilde{p}_j}{2\epsilon^{1/2}}, \tag{98}$$

and

$$\alpha = \frac{1}{8\epsilon^4}, \quad \beta = \frac{\tilde{\beta}}{4}, \quad \gamma = \frac{1}{4\epsilon^4}, \quad \delta = -\frac{1}{8\epsilon^4} + \frac{\tilde{\alpha}}{2\epsilon^2}, \tag{99}$$

and letting $\epsilon \to 0$ while leaving the "renormalized" quantities $\tilde{t}$, etc. finite.

The equations of motion of this model takes the canonical form

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{100}$$

with the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \sum_{j=1}^{\ell} \left[\frac{p_j^2}{2} - \frac{1}{2}\left(\frac{q_j}{2}\right)^6 - 2t\left(\frac{q_j}{2}\right)^4 - 2(t^2-\alpha)\left(\frac{q_j}{2}\right)^2 + \beta\left(\frac{q_j}{2}\right)^{-2} \right] \\ &\quad + g_4^2 \sum_{j \neq k} \left(\frac{1}{(q_j - q_k)^2} + \frac{1}{(q_j + q_k)^2} \right). \end{aligned} \tag{101}$$

The canonical transformation defined by

$$\lambda_j = \left(\frac{q_j}{2}\right)^2, \quad \mu_j = \frac{p_j}{4\sqrt{\lambda_j}} + \frac{1}{4}\left(\lambda_j + 2t + \frac{2\kappa_0}{\lambda_j}\right) \tag{102}$$

maps the foregoing non-autonomous system to the Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \tag{103}$$

with the Hamiltonian

$$H = \sum_{j=1}^{\ell} 2\lambda_j^2 \left[\mu_j^2 - \left(\frac{\lambda_j}{2} + t + \frac{\kappa_0}{\lambda} \right)\mu_j + \frac{\theta_0}{2} \right] + \frac{g_4^2}{4} \sum_{j \neq k} \frac{2(\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k)^2}. \tag{104}$$

V.5.2 Exponential-hyperbolic model and multi-component $\mathrm{P_{III}}$

This model, too, can be derived from the hyperbolic model by degeneration. This degeneration is achieved by the putting the variables and the parameters as

$$q_j = -\tilde{q}_j - \log\frac{\epsilon}{4}, \quad p_j = -\tilde{p}_j, \tag{105}$$

and

$$\alpha = \frac{\tilde{\alpha}}{4\epsilon} + \frac{\tilde{\gamma}}{8\epsilon^2}, \quad \beta = -\frac{\tilde{\gamma}}{8\epsilon^2}, \quad \gamma = \frac{\tilde{\beta}\epsilon}{4}, \quad \delta = \frac{\tilde{\delta}\epsilon^2}{8}, \tag{106}$$

and letting $\epsilon \to 0$.

The equations of motion of this model takes the canonical form

$$t\frac{dq_j}{dt} = \frac{\partial\mathcal{H}}{\partial p_j}, \quad t\frac{dp_j}{dt} = -\frac{\partial\mathcal{H}}{\partial q_j} \tag{107}$$

with the Hamiltonian

$$\begin{aligned}\mathcal{H} \;&=\; \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2} - \frac{\alpha}{4}e^{q_j} + \frac{\beta t}{4}e^{-q_j} - \frac{\gamma}{8}e^{2q_j} + \frac{\delta t^2}{8}e^{-2q_j}\right) \\ &\quad + g_4^2\sum_{j\neq k}\frac{1}{\sinh^2((q_j - q_k)/2)}.\end{aligned} \tag{108}$$

The canonical transformation defined by

$$\lambda_j = e^{q_j}, \quad \mu_j = \frac{p_j}{2\lambda_j} + \frac{1}{2}\left(\eta_\infty + \frac{\theta_0}{\lambda_j} - \frac{\eta_0 t}{\lambda_j^2}\right) \tag{109}$$

maps the foregoing onn-autonomous system to the Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \tag{110}$$

with the Hamiltonian

$$H = \sum_{j=1}^{\ell}\frac{\lambda_j^2}{t}\left[\mu_j^2 - \left(\eta_\infty + \frac{\theta_0}{\lambda_j} - \frac{\eta_0 t}{\lambda_j^2}\right)\mu_j + \frac{\eta_\infty(\theta_0+\theta_\infty)}{2\lambda_j}\right] + \frac{g_4^2}{2t}\sum_{j\neq k}\frac{4\lambda_j\lambda_k}{(\lambda_j-\lambda_k)^2}. \tag{111}$$

V.5.3 Second rational model and multi-component $\mathrm{P_{II}}$

This model can be derived from *both* the rational model and the exponential-hyperbolic model by degeneration. For the degeneration from the rational model, we write the variables and the parameters as

$$t = \frac{-1 + 4^{-1/3}\epsilon^4\tilde{t}}{\epsilon}, \quad \frac{q_j}{2} = \frac{1 + 2^{-1/3}\epsilon^2\tilde{q}_j}{\epsilon^{3/2}}, \quad p_j = \frac{4^{2/3}\tilde{p}_j}{\epsilon^{1/2}} \tag{112}$$

and

$$\alpha = -2\tilde{\alpha} - \frac{1}{2\epsilon^6}, \quad \beta = -\frac{1}{2\epsilon^{12}}, \tag{113}$$

and let $\epsilon \to 0$. The degeneration from the exponential-hyperbolic model is similarly achieved by putting

$$t = 1 + 2\epsilon^2\tilde{t}, \quad q_j = 2\epsilon\tilde{q}_j, \quad p_j = \frac{\tilde{p}_j}{\epsilon}, \tag{114}$$

and

$$\alpha = -\frac{1}{2\epsilon^6}, \quad \beta = \frac{1 + 4\epsilon^3\tilde{\alpha}}{2\epsilon^6}, \quad \gamma = \frac{1}{4\epsilon^6}, \quad \delta = -\frac{1}{4\epsilon^6}, \tag{115}$$

and again letting $\epsilon \to 0$.

The equations of motion of this model takes the canonical form

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{116}$$

with the Hamiltonian

$$\mathcal{H} = \sum_{j=1}^{\ell} \left[\frac{p_j^2}{2} - \frac{1}{2}\left(q_j^2 + \frac{t}{2}\right)^2 - \alpha q_j \right] + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}. \tag{117}$$

The canonical transformation defined by

$$\lambda_j = q_j, \quad \mu_j = p_j + \lambda_j^2 + \frac{t}{2} \tag{118}$$

maps the foregoing non-autonomous system to the Hamiltonian system

$$\frac{d\lambda_j}{dt} = \frac{\partial H}{\partial \mu_j}, \quad \frac{d\mu_j}{dt} = -\frac{\partial H}{\partial \lambda_j} \tag{119}$$

with the Hamiltonian

$$H = \sum_{j=1}^{\ell} \left[\frac{\mu_j^2}{2} - \left(\lambda_j^2 + \frac{t}{2}\right)\mu_j - \left(\alpha + \frac{1}{2}\right)\lambda_j \right] + g_4^2 \sum_{j \neq k} \frac{1}{(\lambda_j - \lambda_k)^2}. \tag{120}$$

V.5.4 Multi-component $\mathrm{P_I}$

This model can be derived from the second rational model, and takes the *same* form on both the Painlevé and Calogero sides. The degeneration process is achieved by putting

$$t = \frac{-6 + \epsilon^{12}\tilde{t}}{\epsilon^{10}}, \quad q_j = \frac{1 + \epsilon^6\tilde{q}_j}{\epsilon^5}, \quad p_j = \frac{\tilde{p}_j}{\epsilon}, \quad \alpha = 4\epsilon^{15} \tag{121}$$

and letting $\epsilon \to 0$. The equations of motion takes the canonical form

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \tag{122}$$

with the Hamiltonian

$$H = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} - 2q_j^3 - tq_j \right) + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}. \tag{123}$$

VI Concluding remarks

We have shown that the Painlevé-Calogero correspondence persists for all the six Painlevé equations and their multi-component generalizations. The Calogero side of this correspondence is a non-autonomous version of Inozemtsev's elliptic model and its various degenerations. Those for $\mathrm{P_V}$ and $\mathrm{P_{IV}}$ are a non-autonomous version of Inozemtsev's hyperbolic and rational models. The others corresponding to $\mathrm{P_{III}}$, $\mathrm{P_{II}}$ and $\mathrm{P_I}$ are further degenerations of the hyperbolic and rational models. The pattern of degeneration on the Calogero side repeats the degeneration diagram

$$\begin{array}{ccccc} \mathrm{P_{VI}} & \longrightarrow & \mathrm{P_V} & \longrightarrow & \mathrm{P_{IV}} \\ & & \downarrow & & \downarrow \\ & & \mathrm{P_{III}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow & \mathrm{P_I} \end{array}$$

of the Painlevé equations.

This picture applies to the autonomous systems as well. Actually, such degeneration relations in the autonomous case have been more or less well known to experts of Calogero-Moser systems (see the Introduction of van Diejen's paper [16]). The autonomous systems are defined by a Hamiltonian of the same form with the time-dependent coupling constants being replaced by absolute constants (except for the elliptic model, in which case an independent time variable is introduced). Those in the position of the first row of the degeneration diagram are, of course, Inozemtsev's elliptic, hyperbolic and rational models (see Section 5). Those in the position of $\mathrm{P_{III}}$ and $\mathrm{P_{II}}$ are defined by the following Hamiltonians:

- Exponential-hyperbolic model:

$$\mathcal{H} = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + g_0^2 e^{q_j} + g_1^2 e^{2q_j} + g_2^2 e^{-q_j} + g_3^2 e^{-2q_j} \right) + g_4^2 \sum_{j \neq k} \frac{1}{\sinh^2((q_j - q_k)/2)}.$$

• Second rational model:

$$\mathcal{H} = \sum_{j=1}^{\ell} \left(\frac{p_j^2}{2} + g_0^2 q_j^4 + g_1^2 q_j^3 + g_2^2 q_j^2 + g_3^2 q_j \right) + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}.$$

The Hamiltonian in the position of $\mathrm{P_I}$ is redundant in the automonous case, because it is a specialization, rather than a degeneration, of the last Hamiltonian.

Note that the Hamiltonian of the second rational model is a *quartic* perturbation of the usual (A_ℓ type) rational Calogero Hamiltonian. According to recent work of Caseiro, Françoise and Sasaki [19], such a quartic (integrable) perturbation always exists for any rational Calogero-Moser system. Inozemtsev's rational model, which is a *sextic* perturbation of the D_ℓ type rational Calogero-Moser system, might admit a similar interpretation.

Back to the Painlevé equations, the extended Painlevé-Calogero correspondence raises many interesting problems. A central issue will be to find an isomonodromic description of the multi-component Painlevé equations. If such an isomonodromic description does exist, it should be related to a new geometric structure.

Acknwlegements

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A Proof of (42)

Let us introduce the two auxiliary functions

$$g(u) = \frac{f_\tau(u)}{f'(u)}, \quad h(u) = \frac{\vartheta'(u+\omega_1)}{\vartheta(u+\omega_1)}, \tag{A.1}$$

associated with the function

$$f(u) = \frac{\wp(u) - e_1}{e_2 - e_1} \tag{A.2}$$

and the standard elliptic theta function

$$\vartheta(u) = \sum_{n=-\infty}^{\infty} \exp(\pi i \tau n^2 + 2\pi i n u). \tag{A.3}$$

Lemma 1 *$g(u)$ is a meromorphic function on the u-plane with additive quasi-periodicity*

$$g(u+1) = g(u), \quad g(u+\tau) = g(u) - 1. \tag{A.4}$$

All poles are of the first order and contained in the lattice $\omega_3 + \mathbb{Z} + \tau\mathbb{Z}$. Furthermore, $g(u)$ has zeros at $u = 0$ and $u = \omega_1$.

Proof. Since $f(u)$ is a doubly periodic function with primitive periods 1 and τ, $f'(u)$ and $f_\tau(u)$ transform as

$$\begin{aligned} f'(u+1) &= f'(u), \qquad f'(u+\tau) = f'(u), \\ f_\tau(u+1) &= f_\tau(u), \qquad f_\tau(u+\tau) = f_\tau(u) - f'(u) \end{aligned}$$

under the shift by 1 and τ. This implies the additive quasi-periodicity of $g(u)$. Furthermore, by the construction, $g(u)$ is a meromorphic function on the u-plane, and all possible poles are of the first order and located at the points of $\omega_k + \mathbb{Z} + \tau\mathbb{Z}$. Let us examine the behavior of $g(u)$ at the representative points $u = \omega_0, \omega_1, -\omega_2, \omega_3$:

- As $u \to \omega_0 = 0$,

  $$f(u) = \frac{1}{(e_2 - e_1)u^2} + O(1),$$

  thereby

  $$f'(u) = -\frac{2}{(e_2 - e_1)u^3} + O(1), \quad f_\tau(u) = -\frac{e_{2,\tau} - e_{1,\tau}}{(e_2 - e_1)^2 u^2} + O(1),$$

  so that $g(u)$ has rather a zero at $u = 0$:

  $$g(u) = O(u). \tag{A.5}$$

- As $u \to \omega_1 = \frac{1}{2}$,

  $$\begin{aligned} f(u) &= \frac{1}{e_2 - e_1}\Big(\wp(\omega_1) - e_1 + \wp'(\omega_1)(u - \omega_1) + O((u-\omega_1)^2)\Big) \\ &= O((u-\omega_1)^2), \end{aligned}$$

  thereby

  $$f'(u) = O(u - \omega_1), \quad f_\tau(u) = O((u-\omega_1)^2),$$

  so that $g(u)$ has another zero at $u = \omega_1$:

  $$g(u) = O(u - \omega_1). \tag{A.6}$$

- As $u \to -\omega_2 = \frac{1}{2} + \frac{\tau}{2}$,

$$\begin{aligned} f(u) &= \frac{1}{e_2 - e_1}\Big(\wp(-\omega_2) - e_1 + \wp'(-\omega_2)(u + \omega_2) + O((u + \omega_2)^2)\Big) \\ &= O((u + \omega_2)^2), \end{aligned}$$

thereby

$$f'(u) = O(u + \omega_2), \quad f_\tau(u) = O(u + \omega_2),$$

so that $g(u)$ behaves as

$$g(u) = O(1). \tag{A.7}$$

- As $u \to \omega_3 = \frac{\tau}{2}$,

$$\begin{aligned} f(u) &= \frac{1}{e_2 - e_1}\Big(\wp(\omega_3) - e_1 + \wp'(\omega_3)(u - \omega_3) + O((u - \omega_3)^2)\Big) \\ &= t + O((u - \omega_3)^2), \end{aligned}$$

thereby

$$f'(u) = O(u - \omega_3), \quad f_\tau(u) = O(1),$$

so that $g(u)$ turns out to have a pole of the first order at $u = \omega_3$:

$$g(u) = 0((u - \omega_3)^{-1}). \tag{A.8}$$

The behavior of $g(u)$ at the other points of $\omega_n + \mathbb{Z} + \tau\mathbb{Z}$ can be deduced from these results by the additive quasi-periodicity of $g(u)$. *Q.E.D.*

Lemma 2 *$h(u)$ is a meromorphic function on the u-plane with additive quasi-periodicity*

$$h(u + 1) = h(u), \quad h(u + \tau) = h(u) - 2\pi i. \tag{A.9}$$

All poles are of the first order and contained in the lattice $\omega_3 + \mathbb{Z} + \tau\mathbb{Z}$. Furthermore, $h(u)$ has zeros at $u = 0$ and $u = \omega_1$.

Proof. Let us recall the fundamental properties of $\vartheta(u)$:

- $\vartheta(u)$ is an entire function on the u-plane with zeros of the first order at the lattice points $\omega_2 + m + n\tau$ $(m, n \in \mathbb{Z})$.

- $\vartheta(u)$ is quasi-periodic,

$$\vartheta(u+1) = \vartheta(u), \quad \vartheta(u+\tau) = e^{-\pi i\tau - 2\pi i u}\vartheta(u).$$

- $\theta(u)$ and $\vartheta(u+1/2)$ are even under the reflection $u \to -u$.

All the properties of $h(u)$ in the statement of the lemma are an immediate consequence of these properties of $\vartheta(u)$. *Q.E.D.*

Lemma 3 *The function $f(u)$ satisfies the equation*

$$2\pi i \frac{f_\tau(u)}{f'(u)} = \frac{\vartheta'(u+\omega_1)}{\vartheta(u+\omega_1)}, \tag{A.10}$$

where the prime stands for $\partial/\partial u$.

Proof. The foregoing properties of $g(u)$ and $h(u)$ imply the following:

- $2\pi i g(u) - h(u)$ is a doubly periodic meromorphic function with fundamental period 1 and τ.

- All poles of $2\pi i g(u) - h(u)$ are of the first order and contained in the lattice $\omega_3 + \mathbb{Z} + \tau\mathbb{Z}$.

- $2\pi i g(u) - h(u)$ has zeros at $u = 0$ and $u = \omega_1$.

The first two properties imply that $2\pi i g(u) - h(u)$ is a constant. By the last one, this constant has to be zero. We thus find that $2\pi i g(u) - h(u) = 0$. *Q.E.D.*

Lemma 4 *$\vartheta(u)$ satisfies the equation*

$$\Big(\log \vartheta(u+\omega_1)\Big)'' = -\wp(u+\omega_3) + \text{function of } \tau \text{ only}. \tag{A.11}$$

Proof. The aforementioned complex analytic properties of $\vartheta(u)$ imply the following:

- $\Big(\log\vartheta(u+\omega_1)\Big)''$ is a doubly periodic meromorphic function with primitive period 1 and τ.
- All poles of this meromorphic function are contained in the lattice $\omega_3+\mathbb{Z}+\tau\mathbb{Z}$.
- As $u\to-\omega_3$, this function behaves as

$$\Big(\log\vartheta(u+\omega_1)\Big)''=-\frac{1}{(u+\omega_3)^2}+O(1).$$

The function $-\wp(u+\omega_3)$, too, has these properties. Accordingly, their difference is a constant function on the u-plane, namely, a function of τ only. *Q.E.D.*

We now return to the proof of (42). By the third lemma, we have the identity

$$2\pi i\frac{f_\tau(u)}{f'(u)}du=\frac{\vartheta'(u+\omega_1)}{\vartheta(u+\omega_1)}du=\frac{d\vartheta(u+\omega_1)}{\vartheta(u+\omega_1)}-\frac{\partial\vartheta(u+\omega_1)/\partial\tau}{\vartheta(u+\omega_1)}d\tau \tag{A.12}$$

On the other hand, the well known "heat equation"

$$4\pi i\frac{\partial\vartheta(u)}{\partial\tau}=\vartheta(u)'' \tag{A.13}$$

implies that

$$\frac{\partial\vartheta(u+\omega_1)/\partial\tau}{\vartheta(u+\omega_1)}=\frac{1}{4\pi}\frac{\vartheta(u+\omega_1)''}{\vartheta(u+\omega_2)}=\frac{1}{4\pi i}\left[\Big(\log\vartheta(u+\omega_1)\Big)''+\left(\frac{\vartheta'(u+\omega_1)}{\vartheta(u+\omega_1)}\right)^2\right].$$

By the third and forth lemmas, the last line can be rewritten

$$\frac{1}{4\pi i}\left[-\wp(u+\omega_3)+\left(2\pi i\frac{f_\tau(u)}{f'(u)}\right)^2\right]+\text{function of }\tau\text{ only},$$

so that

$$2\pi i\frac{f_\tau(u)}{f'(u)}du=\frac{1}{4\pi i}\left[\wp(u+\omega_3)-\left(2\pi i\frac{f_\tau(u)}{f'(u)}\right)^2\right]d\tau+\text{exact form}. \tag{A.14}$$

Substituting $u=q$ gives (42)

B Asymptotics of elliptic functions

The asymptotic behavior of the $\wp$-function $\wp(u)$, the shifted $\wp$-functions $\wp(u+\omega_k)$ and the constants $e_k = \wp(\omega_k)$, in the limit as $\operatorname{Im}\tau \to +\infty$, can be deduced from the well known formula

$$\wp(u) = \sum_{n=-\infty}^{\infty} \frac{\pi^2}{\sin^2(\pi(u+n\tau))} - \frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{2\pi^2}{\sin^2(\pi n\tau)}. \tag{B.1}$$

Let us first consider the asymptotic behavior of $\wp(u)$ itself. The constant $(n=0)$ term in the first sum is of order 1 and the n-th term is of order $e^{2n\pi i\tau}$. Similarly, the n-th term in the second sum is of order $e^{2n\pi i\tau}$. Therefore

$$\wp(u) = \frac{\pi^2}{\sin^2(\pi u)} - \frac{\pi^2}{3} + O(e^{2\pi i\tau}). \tag{B.2}$$

A similar estimate leads to the following asymptotic expression for the shifted $\wp$-functions:

$$\begin{aligned}
\wp(u+\omega_1) &= \frac{\pi^2}{\cos^2(\pi u)} - \frac{\pi^2}{3} + O(e^{2\pi i\tau}),\\
\wp(u+\omega_2) &= -\frac{\pi^2}{3} + 8\pi^2\cos(2\pi u)e^{\pi i\tau} + O(e^{2\pi i\tau}),\\
\wp(u+\omega_3) &= -\frac{\pi^2}{3} - 8\pi^2\cos(2\pi u)e^{2\pi i\tau} + O(e^{2\pi i\tau}).
\end{aligned} \tag{B.3}$$

In fact, the degeneration process of the elliptic model requires us to know the asymptotic expression of $\wp(u+\omega_2)+\wp(u+\omega_3)$ to the order $e^{2\pi i\tau}$. This can be achieved by the following calculations:

$$\begin{aligned}
&\wp(u+\omega_2)+\wp(u+\omega_3)\\
&= \sum_{n=-\infty}^{\infty} \frac{\pi^2}{\cos^2(u+\frac{\tau}{2}+n\tau)\sin^2(u+\frac{\tau}{2}+n\tau)} - \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4\pi^2}{\sin^2(\pi n\tau)}\\
&= -\frac{2\pi^2}{3} - 32\pi^2\cos(2\pi u)e^{2\pi i\tau} + 16\pi^2 e^{2\pi i\tau} + O(e^{3\pi i\tau}).
\end{aligned} \tag{B.4}$$

We now consider the constants e_k. For instance, e_1 can be written

$$\begin{aligned}
e_1 &= \sum_{n=-\infty}^{\infty} \frac{\pi^2}{\cos^2(\pi n\tau)} - \frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{2\pi^2}{\sin^2(\pi n\tau)}\\
&= \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2\pi^2}{\cos^2(\pi n\tau)} - \sum_{n=1}^{\infty}\frac{2\pi^2}{\sin^2(\pi n\tau)}.
\end{aligned} \tag{B.5}$$

The constant $2\pi^2/3$ becomes the leading term; the leading $(n = 1)$ terms of the last two series give the next-leading term of the order $e^{2\pi i\tau}$. e_2 and e_3 can be similarly analyzed. Thus the following asymptotic formulas are obtained:

$$\begin{aligned}
e_1 &= \frac{2\pi^2}{3} + 16\pi^2 e^{2\pi i\tau} + O(e^{4\pi i\tau}), \\
e_2 &= -\frac{\pi^2}{3} + 8\pi^2 e^{\pi i\tau} + O(e^{2\pi i\tau}) \\
e_3 &= -\frac{\pi^2}{3} - 8\pi^2 e^{\pi i\tau} + O(e^{2\pi i\tau}).
\end{aligned} \tag{B.6}$$

In particular, $e_2 - e_1 \to -\pi^2$, as expected.

References

[1] P. Painlevé, Memoire sur les équations différentielles dont l'intégrale générale est uniforme, Bull. Soc. Math. Phys. France **28** (1900), 201-261; Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme, Acta Math. **21** (1902), 1-85.

[2] B. Gambier, Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critique fixés, C.R. Acad. Sci. (Paris) **142** (1906), 266-269; ditto, Acta. Math. **33** (1910), 1-55.

[3] R. Fuchs, Sur quelques équations différentielles linéaires du second ordre, C.R. Acad. Sci. (Paris) **141** (1905), 555-588; Über lineare homogene Differentialgleichungern zweiter Ordnung mit im endlich gelegene wesentlich singulären Stellen, Math. Ann. **63** (1907), 301-321.

[4] P. Painlevé, Sur les équations différentielles du second ordre à points critiques fixés, C.R. Acad. Sci. (Paris) **143** (1906), 1111-1117.

[5] K. Okamoto, Studies on the Painlevé equations, I: Sixth Painlevé equation, Annali Mat. Pura Appl. **146** (1987), 337-381.

[6] Yu. I. Manin, Sixth Painlevé equation, universal elliptic curve, and mirror of $\mathbb{P}^2$, AMS Transl. (2) **186** (1998), 131-151.

[7] A.M. Levin and M.A. Olshanetsky, Painlevé-Calogero correspondence, e-print `alg-geom/9706012`; Classical limit of the Knizhnik-Zamolodchikov-Bernard equations as hierarchy of isomonodromic deformations, e-print `hep-th/9709207`.

[8] M.A. Olshanetsky and A.M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Physics Reports **71** (1981), 313-400.

[9] F. Calogero, Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials, J. Math. Phys. **12** (1971) 419-436; Exactly solvable one-dimensional many body problems, Lett. Nuovo Cim. **13** (1975) 411-416.

[10] V.I. Inozemtsev and D.V. Meshcheryakov, Extension of the class of integrable dynamical systems connected with semisimple Lie algebras, Lett. Math. Phys. **9** (1985), 13-18.

[11] V.I. Inozemtsev, Lax representation with spectral parameter on a torus for integrable particle systems, Lett. Math. Phys. **17** (1989), 11-17.

[12] M.A. Olshanetsky, Painlevé type equations and Hitchin systems, e-print `math-ph/9904023`.

[13] K. Okamoto, Isomonodromic deformations and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **33** (1986), 575-618.

[14] D. Levi and S. Wojciechowski, On the Olshanetsky-Perelomov many-body system in an external field, Phys. Lett. **A103** (1984), 11-14.

[15] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, From Gauss to Painlevé (Vieweg, Braunschweig, 1991).

[16] J.F. van Diejen, Difference Calogero-Moser systems and finite Toda chains, J. Math. Phys. **36** (1995), 1299-1323.

[17] K. Takasaki, Elliptic calogero-Moser systems and isomonodromic deformations, J. Math. Phys. 40 (1999), 5787-5821.

[18] J. Malmquist, Sur les équations différentielles du second ordre dont l'intégrale générale a ses points critique fixes, Ark. Mat. Astr. Fys. **17** (1922/23), 1-89.

[19] R. Caseiro, J.-P. Françoise and R. Sasaki, Algebraic linearization of dynamics of Calogero type for any coxeter group, e-print `hep-th/0001074`.